Math 211 - Bonus Exercise 11 (please discuss on Forum)

1) Prove that a finite nilpotent group G has a normal subgroup of any order dividing $|G|$. Hint: first prove the result for p-groups.

2) Prove that D_{2n} is nilpotent if and only if n is a power of 2.

3) (Frattini's argument): if G is finite and $H \trianglelefteq G$ is a normal subgroup, let P be a Sylow p-subgroup of H. Show that G is generated by H and $N_G(P)$, i.e. $G = HN_G(P)$.

4) For any finite group G, show that its Frattini subgroup

$$\Phi(G) = \bigcap_{H<G \text{ maximal}} H$$

is nilpotent (a subgroup is called maximal if it is not contained in any other subgroup except G). Hint: prove that the Frattini subgroup is normal in G, and then use the previous exercise.